Correction TD1 : Transformée de Laplace

$$F_1(p) = \frac{p+10}{p(p^2+23p+120)}$$

$$\Delta = 23^2 - 4(120) = 49 \quad \begin{cases} p_1 = \frac{-23+\sqrt{49}}{2} = -8 \\ p_2 = \frac{-23-\sqrt{49}}{2} = -15 \end{cases}$$

Donc $F_1(p) = \frac{p+10}{p(p+8)(p+15)} = \frac{A}{p} + \frac{B}{p+8} + \frac{C}{p+15}$

On calcule les coefficients A, B et C

$$A = p\,F_1(p)\Big|_{p=0} = \frac{p+10}{(p+8)(p+15)}\Big|_{p=0} = \frac{10}{8\times 15} = \frac{1}{12}$$

$$B = (p+8)\,F_1(p)\Big|_{p=-8} = \frac{p+10}{p(p+15)}\Big|_{p=-8} = \frac{-8+10}{-8(-8+15)}$$

$$B = \frac{2}{-8\times 7} = \frac{-2}{56} = \frac{-1}{28}$$

$$C = (p+15)\,F_1(p)\Big|_{p=-15} = \frac{p+10}{p(p+8)}\Big|_{p=-15} = \frac{-15+10}{-15(-15+8)}$$

$$C = \frac{-5}{-15\times -7} = \frac{1}{21}$$

Donc $F_1(p) = \frac{1/12}{p} + \frac{-1/28}{p+8} + \frac{1/21}{p+15}$

D'après la table : $f_1(t) = \left(\frac{1}{12} - \frac{1}{28}e^{-8t} + \frac{1}{21}e^{-15t}\right)$

$$\boxed{f_1(t) = \left(\frac{1}{12} - \frac{e^{-8t}}{28} + \frac{e^{-15t}}{21}\right)u(t)}$$

Vérification $\lim\limits_{t\to+\infty} f_1(t) = 1/12$

$$f_1(+\infty) = \lim_{p\to 0} p\,F_1(p) = \frac{p+10}{p^2+23p+120}\Big|_{p=0} = \frac{1}{12}$$

$$F_2(p) = \frac{16p}{(p+16)(p^2+23p+60)}$$

on cherche les racines réells :

$$\Delta = 23^2 - 4(60) = 289 = 17^2$$

$$p_1 = \frac{-23-\sqrt{289}}{2} = -20$$

$$p_2 = \frac{-23+\sqrt{289}}{2} = -3$$

Donc on peut décomposer $\underbrace{\frac{1}{p^2+23p+60}}_{(p+3)(p+20)}$ en éléments simples.

$$F_2(p) = \frac{A}{(p+16)} + \frac{B}{(p+3)} + \frac{C}{(p+20)}$$

$$A = F_2(p) \times (p+16)\Big|_{p=-16} = \frac{16p}{(p+3)(p+20)}\Big|_{p=-16}$$

$$A = \frac{-16\times16}{(-16+3)(-16+20)} = \frac{-256}{-13\times4} = \frac{64}{13}$$

$$B = (p+3)\,F_2(p)\Big|_{p=-3} = \frac{16p}{(p+16)(p+20)}\Big|_{p=-3} = \frac{-48}{13\times17}$$

$$B = \frac{-48}{221}$$

$$C = (p+20)F_2(p)\Big|_{p=-20} = \frac{16p}{(p+16)(p+3)}\Big|_{p=-20}$$

$$C = \frac{+16\times(-20)}{(-4)\times(-17)} = \frac{-320}{4\times17} = \frac{-80}{17}$$

$$\text{Donc } F_2(p) = \frac{64/13}{p+16} + \frac{-48/221}{p+3} + \frac{-80/17}{p+20}$$

$$f_2(t) = \frac{64}{13}e^{-16t} - \frac{48}{221}e^{-3t} - \frac{80}{17}e^{-20t}$$

vérification

- calcul de la valeur initiale :

$$f_2(0) = \lim_{p \to +\infty} p F_2(p) = \lim_{p \to +\infty} p \times \frac{16p}{(p+16)(p^2+13p+60)} = 0$$

on calcule $f_2(0) = \frac{64}{13} - \frac{48}{221} - \frac{80}{17} = \underline{0}$

* $F_3(p) = \frac{10}{p^2+10p+34}$

$\Delta = 10^2 - 4 \times 34 = -36 < 0 \Rightarrow$ pas de racines réelles.

$$p^2+10p+34 = \underbrace{p^2 + \underbrace{2 \times 5}_{10} p + 25}_{(p+5)^2} + \underbrace{9}_{34} = (p+5)^2 + 3^2$$

Donc $F_3(p) = \frac{10}{(p+5)^2 + 3^2} = \frac{10}{3} \times \underbrace{\frac{3}{(p+5)^2+3^2}}$

TL $\sin at \longrightarrow \frac{a}{p^2+a^2}$

Donc $f_3(t) = \frac{10}{3} \times \sin 3t \times \underbrace{e^{-5t}}_{\text{retard}} \times u(t)$

$(F(p+a) \longrightarrow e^{-at} f(t))$

$$\boxed{f_3(t) = \frac{10}{3} \cdot \sin(3t) \cdot e^{-5t}\, u(t)}$$

$$F_4(p) = \frac{12p}{p^2+16p+100}$$

$\Delta = 16^2 - 4\times100 = -144 < 0 \Rightarrow$ pas de racines reelles.

$$p^2+16p+100 = \underbrace{p^2+2\times8p+64}_{(p+8)^2} + 36 \quad (64+36 = 100)$$

Donc $F_4(p) = \dfrac{12p}{(p+8)^2+36} = \dfrac{12p}{(p+8)^2+6^2}$

on a la forme d'un cosinus retardé de 8.

$$TL^{-1}\left(\frac{p}{p^2+k^2}\right) \longrightarrow \cos kt\, U(t)$$

$$TL^{-1}\left(\frac{p+a}{(p+a)^2+k^2}\right) \longrightarrow \cos kt.\, e^{-at}$$

Donc $F_4(p) = 12.\dfrac{p}{(p+8)^2+6^2} = 12\dfrac{p+8-8}{(p+8)^2+6^2}$

$$F_4(p) = 12.\left(\frac{p+8}{(p+8)^2+6^2} - \frac{8}{(p+8)^2+6^2}\right)$$

$$F_4(p) = 12\left[\underbrace{\frac{p+8}{(p+8)^2+6^2}}_{\cos} - \frac{8}{6}\times\underbrace{\frac{6}{(p+8)^2+6^2}}_{\sin}\right]$$

$$f_4(t) = 12\left[\cos 6t.e^{-8t} - \frac{8}{6}\times \sin 6t.e^{-8t}\right]$$

$$\boxed{f_4(t) = 12\left[\cos 6t - \frac{4}{3}\sin 6t\right]e^{-8t}.U(t)}$$