(1)

Correction TD2 : Transformée en z

Exercice:

a) $F(z) = \dfrac{3z}{3z^2 - 2z - 1}$

$\Delta = (-2)^2 - 4(-1)\times 3 = 4 + 12 = 16$

$z_1 = \dfrac{+2 - \sqrt{16}}{6} = -\dfrac{2}{6} = -1/3$

$z_2 = \dfrac{+2 + \sqrt{16}}{6} = 1$

$F(z) = \dfrac{3z}{(z + \frac{1}{3})(z-1)}$

$\dfrac{F(z)}{z} = \dfrac{3}{(z+\frac{1}{3})(z-1)} = \dfrac{A}{z+\frac{1}{3}} + \dfrac{B}{z-1}$

$A = \dfrac{F(z)}{z} \times (z + \frac{1}{3}) \Big|_{z=-1/3} = \dfrac{3}{z-1}\Big|_{z=-1/3} = \dfrac{3}{-1/3 - 1}$

$\boxed{A = -9/4}$

$B = \dfrac{F(z)}{z} \times (z-1)\Big|_{z=1} = \dfrac{3}{z + 1/3}\Big|_{z=1} = \dfrac{3}{4/3} = \dfrac{9}{4}$

ce qui fait : $F(z) = \dfrac{-9}{4} \cdot \dfrac{z}{z + 1/3} + \dfrac{9}{4} \dfrac{z}{z-1}$

$f(n) = \left(-\dfrac{9}{4}\left(-\dfrac{1}{3}\right)^n + \dfrac{9}{4} \times 1\right) u(n)$

$$\boxed{f(n) = \frac{9}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right)}$$

b) $F(z) = \dfrac{4z^2}{(z+5)^2}$ on utilise le théorème des résidus

on pose $H(z) = z^{n-1} \cdot F(z) = z^{n-1} \cdot \dfrac{\times 4z^2}{(z+5)^2}$

$\Rightarrow H(z) = 4 \cdot \dfrac{z^{n+1}}{(z+5)^2}$ on a un pôle double en $z = -5$

$\text{Res}(H(z), -5) = f(n) = \dfrac{d}{dz}\left[(z+5)^2 H(z)\right]\Big|_{z=-5} = \dfrac{d}{dz}(4 \cdot z^{n+1})\Big|_{z=-5}$

$\Rightarrow \boxed{f(n) = 4(n+1) \times (-5)^n}$

c) $F(z) = \frac{2}{z^2+10z+24} = \frac{2}{(z+4)(z+6)}$

méthode des résidus:

on pose $H(z) = z^{n-1} \cdot F(z) = \frac{2z^{n-1}}{(z+4)(z+6)}$

pour $n=0 \Rightarrow H(z)$ possède un pôle simple

$H(z) = \frac{2}{z(z+4)(z+6)}$

$Res(H(z),0) = z \times F(z)\Big|_{z=0} = \frac{2}{(z+4)(z+6)}\Big|_{z=0} = \frac{1}{12}$

$Res(H(z),-4) = (z+4)\,H(z)\Big|_{z=-4} = \frac{2}{z(z+6)}\Big|_{z=-4}$

$Res(H(z),-4) = \frac{2}{-4\times 2} = -\frac{1}{4}$

$Res(H(z),-6) = (z+6)\,H(z)\Big|_{z=-6} = \frac{2}{z(z+4)}\Big|_{z=-6}$

$Res(H(z),-6) = \frac{2}{-6\times(-2)} = +\frac{1}{6}$

$x_0 = \frac{1}{12} - \frac{1}{4} + \frac{1}{6} = 0$

Pour $n \geq 1$ $\quad H(z) = \frac{2 \cdot z^{n-1}}{(z+4)(z+6)}$

on a: $Res(H(z),-4) = \frac{2 \cdot z^{n-1}}{z+6}\Big|_{z=-4}$

$Res(H(z),-4) = \frac{2\times(-4)^{n-1}}{2} = (-4)^{n-1}$

et $Res(H(z),-6) = \frac{2z^{n-1}\times(z+6)}{(z+4)(z+6)}\Big|_{z=-6} = \frac{2z^{n-1}}{z+4}\Big|_{z=-6}$

$R(H(z),-6) = \frac{2(-6)^{n-1}}{-6+4} = -(-6)^{n-1}$

donc $\boxed{f(n) = (-4)^{n-1} - (-6)^{n-1}}$

d) $F(z) = \dfrac{5}{6 - 5z^{-1} + z^{-2}} = \dfrac{5z^2}{6z^2 - 5z + 1}$

$\Delta = (-5)^2 - 4(1)(6) = 1$

$z_1 = \dfrac{5+\sqrt{1}}{12} = \dfrac{1}{2}$ et $z_2 = \dfrac{5-\sqrt{1}}{12} = \dfrac{1}{3}$

$F(z) = \dfrac{5z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}$

$F(z) = z \cdot \dfrac{5z}{(z-\frac{1}{2})(z-\frac{1}{3})}$

$F_1(z) = \dfrac{5z}{(z-\frac{1}{2})(z-\frac{1}{3})} = \dfrac{A}{z-\frac{1}{2}} + \dfrac{B}{z-1/3}$

$A = F_1(z) \times (z-\frac{1}{2})\Big|_{z=\frac{1}{2}} = \dfrac{5z}{z-\frac{1}{3}}\Big|_{z=1/2} = \dfrac{5/2}{1/2-1/3}$

$A = \dfrac{5/2}{1/6} = \dfrac{5\times 6}{2} = 15$

$B = F_1(z) \cdot (z-\frac{1}{3})\Big|_{z=1/3} = \dfrac{5z}{z-\frac{1}{2}}\Big|_{z=\frac{1}{3}} = \dfrac{5/3}{\frac{1}{3}-\frac{1}{2}}$

$B = \dfrac{5/3}{-\frac{1}{6}} = -\dfrac{5\times 6}{3} = -10$

donc $F(z) = \dfrac{15z}{z-1/2} - \dfrac{10z}{z-\frac{1}{3}}$

$\downarrow$

$\boxed{f(n) = 15 \times (\frac{1}{2})^n - 10 \times (\frac{1}{3})^n}$

méthode : des Residus

Soit $H(z) = z^{n-1} \cdot F(z) = z^{n-1} \dfrac{5z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}$

$$H(z) = \frac{5z^{n+1}}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$\mathrm{Res}\left(H(z), \tfrac{1}{2}\right) = \left.\frac{5z^{n+1}}{z-\frac{1}{3}}\right|_{z=\frac{1}{2}} = \frac{5\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}-\frac{1}{3}} = 30\left(\frac{1}{2}\right)^{n+1}$$

$$\mathrm{Res}\left(H(z), \tfrac{1}{3}\right) = \left.\frac{5z^{n+1}}{z-\frac{1}{2}}\right|_{z=\frac{1}{3}} = \frac{5z^{n+1}}{\frac{1}{3}-\frac{1}{2}} = \frac{5\cdot\ {}^{n+1}}{\frac{2-3}{6}} = -30\left(\frac{1}{3}\right)^{n+1}$$

donc $f(n) = 30 \cdot \left(\frac{1}{2}\right)^{n+1} - 30\left(\frac{1}{3}\right)^{n+1}$

$$\boxed{f(n) = 30 \cdot \left(\frac{1}{2}\right)^{n+1} - 30\left(\frac{1}{3}\right)^{n+1}}$$

n	0	1	2	3
2ème méthode $f(n)$	5	$\frac{15}{2}-\frac{10}{3}$	$\frac{15}{4}-\frac{10}{9}$	$\frac{15}{8}-\frac{10}{27}$
1ère méthode $f(n)$	5	$\frac{15}{2}-\frac{10}{3}$	$\frac{15}{4}-\frac{10}{9}$	$\frac{15}{8}-\frac{10}{27}$

On trouve exactement les mêmes valeurs pour les 2 suites.